Realization of fractional order controllers

***Igor Podlubny*[1], *Ivo Petras*[1], *Blas M. Vinagre*[2], *YangQuan Chen*[3], *Paul O'Leary*[4]** *and* ***Lubomir Dorcak*[1]**

Realizácia regulátorov neceločíselného rádu

An approach to realizations of fractional-order controllers is presented. The suggested approach is based on the use of continued fraction expansions.

Key words: *fractional calculus, fractional differentiation, fractional integration, fractional order controller, realization*.

Fractional-Order Systems and Controllers

General information about various approaches to fractional-order differentiation and integration can be found in the available monographs on this subject (Kiryakova, 1994; Podlubny, 1999; Samko, Kilbas and Matichev, 1987). Because of this, we do not discuss general definitions here. Instead, we recall only the expressions for describing fractional-order systems and $PI^{\lambda}D^{\mu}$ controllers (Podlubny, 1999).

A wide class of linear fractional-order control systems can be described by fractional differential equations of the form

$$a_n D^{\alpha_n} y(t) + a_{n-1} D^{\alpha_{n-1}} y(t) + \ldots + a_0 D^{\alpha_0} y(t) =$$
$$= b_m D^{\beta_m} u(t) + b_{m-1} D^{\beta_{m-1}} u(t) + \ldots + b_0 D^{\beta_0} u(t), \qquad (1)$$

or by continuous transfer functions of the form:

$$G(s) = b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + \ldots + b_0 s^{\beta_0} a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \ldots + a_0 s^{\alpha_0}, \qquad (2)$$

where: $D^{\gamma} \equiv {}_0D_t^{\gamma}$ denotes the Riemann-Liouville or Caputo fractional derivative (Podlubny, 1999); a_k $(k = 0,\ldots n)$, b_k $(k = 0,\ldots m)$ are constant; and α_k $(k = 0,\ldots n)$, β_k $(k = 0,\ldots m)$ are arbitrary real numbers. Without loss of generality we can assume that $\alpha_n > \alpha_{n-1} > \ldots > \alpha_0$, and $\beta_m > \beta_{m-1} > \ldots > \beta_0$.

The fractional-order $PI^{\lambda}D^{\mu}$ controller was proposed (Podlubny, 1999) as a generalization of the PID controller with integrator of real order λ and differentiator of real order μ. The transfer function of such type the controller in Laplace domain has the form:

$$G_c(s) = \frac{U(s)}{E(s)} = K + T_i s^{-\lambda} + T_d s^{\mu}, (\lambda, \mu > 0), \qquad (3)$$

where: K is the proportional constant, T_i is the integration constant and T_d is the differentiation constant. Taking $\lambda = 1$ and $\mu = 1$, we obtain a classical PID controller. If $\lambda = 0$ and/or $T_i = 0$, we obtain a PD^{μ} controller, etc.

Continued fractions and multiple loops

Recently we have established an interesting new relationship between continued fractions and nested multiple-loop control systems. Namely, the transfer function of the nested multiple-loop control system shown in Figure 1 has the form of the following continued fraction expansion:

[1] *Igor Podlubny, Ivo Petras, Lubomir Dorcak:* Department of Applied Informatics and Process Control, BERG Faculty, Technical University of Kosice, Slovak Republic

[2] *Blas M. Vinagre:* School of Industrial Engineering, University of Extramadura, Spain

[3] *YangQuan Chen:* Department of Electrical and Computer Engineering, Utah State University, USA

[4] *Paul O'Leary:* Institute of Automation, Montanuniversitat Leoben, Austria

(Recenzovaná a revidovaná verzia dodaná 19.11.2003)

$$Z(s) = Z_1(s) + \cfrac{1}{Y_2(s) + \cfrac{1}{Z_3(s) + \cfrac{1}{Y_4(s) + \cfrac{1}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots}}}} \qquad (4)$$

$$\cfrac{1}{Y_{2n-2}(s) + \cfrac{1}{Z_{2n-1}(s) + \cfrac{1}{Y_{2n}(s)}}}$$

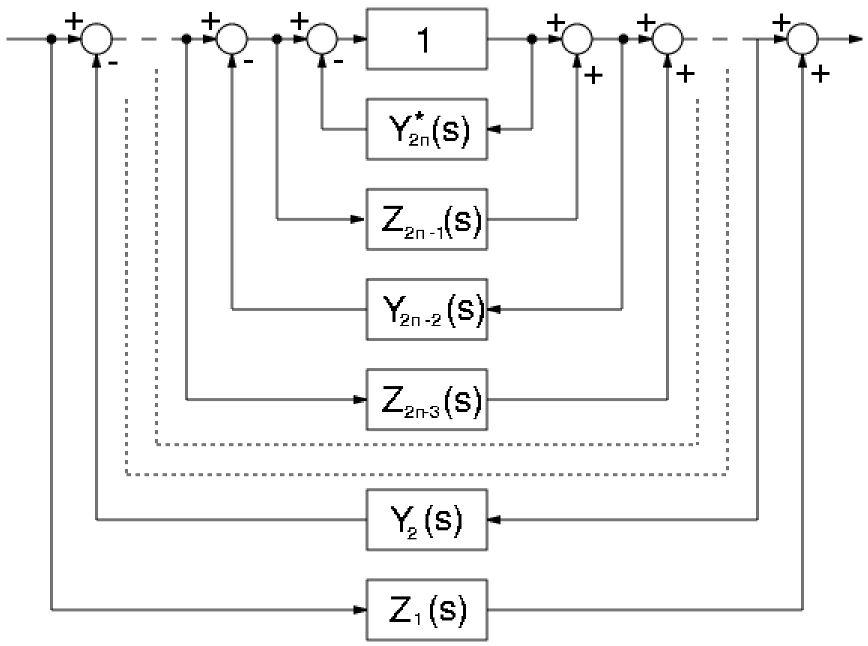

Fig.1. Nested multiple-loop control system of the first type.

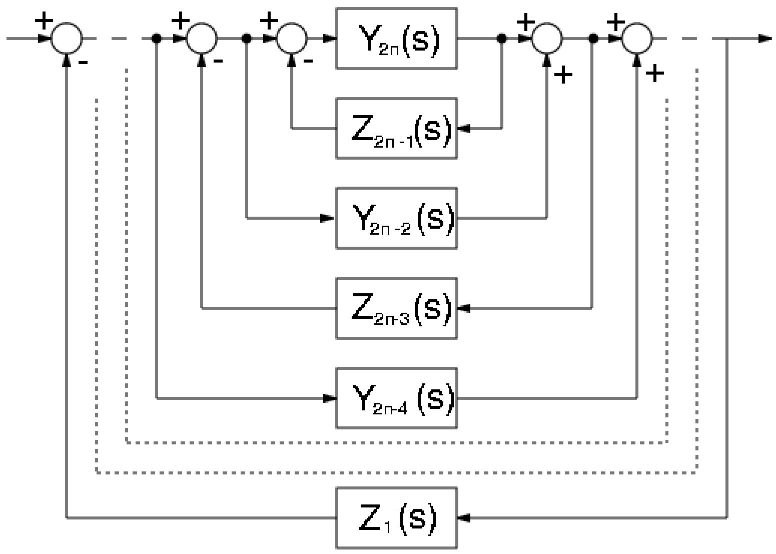

Fig.2. Nested multiple-loop control system of the second type.

Similarly, the continued fraction expansion of the transfer function of the other useful type of a nested multiple-loop control system, depicted in Figure 2, is:

$$Z(s)=\cfrac{1}{Z_1(s)+\cfrac{1}{Y_2(s)+\cfrac{1}{Z_3(s)+\cfrac{1}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cfrac{1}{Y_{2n-2}(s)+\cfrac{1}{Z_{2n-1}(s)+\cfrac{1}{Y_{2n}(s)}}}}}}} \tag{5}$$

Both these types of nested multiple-loop systems can be used for simulations and realizations of arbitrary transcendental transfer functions. For this, the transfer function should be developed in a continued fraction, which after truncation can be represented by a nested multiple-loop system shown in Figure 1 or Figure 2.

Negative impedances

Continued fraction expansions provide a general tool for various realizations of fractional order controllers, both analogue and digital. The case of negative coefficients in the continued fraction expansion is of special interest from the viewpoint of realization in the form of Cauer's canonic circuits (Kvasil, 1981), when one has therefore to deal with negative impedances.

The possibility of realization of negative impedances in electric circuits has been pointed out by Bode (Bode, 1949, Chapter IX). Later, in 1970s, operational amplifiers appeared, which significantly simplified creation of circuits exhibiting negative resistances, negative capacitances, and negative inductances. Such circuits are called *negative-impedance converters* (see sample circuits in (Dostal, 1993)).

Conclusion

The use of continued fraction expansions is a good general method for designing devices (fractances) described by fractional differential equations or by fractional-order transfer functions. Moreover, this approach can be used for realization of other types of systems with transcendental transfer functions, which can be developed in continued fractions.

The two types of nested multiple-loop systems, described in this paper, can be used for modelling, simulation, and realization of fractional-order systems and controllers, and, in general, for modelling, simulation and realization of systems with known rational approximation of their transfer function.

References

BODE, H. W.: Network Analysis and Feedback Amplifier Design, *Tung Hwa Book Co*., Shanghai, China, 1949.

DOSTAL, J.: Operational Amplifiers, *Butterworth–Heinemann*, Boston, 1993.

KIRYAKOVA, V.: Generalized Fractional Calculus and Applications. *Longman*, New York, 1994.

KVASIL, J. and ČAJKA, J.: An Introduction to Synthesis of Linear Circuits, *SNTL/ALFA*, Prague, 1981. (in Czech)

PODLUBNY, I.: Fractional Differential Equations. *Acad. Press*, San Diego, 1999.

SAMKO, S. G., KILBAS, A. A., and MARITCHEV, O. I.: Integrals and Derivatives of the Fractional Order and Some of Their Applications, *Nauka i Tekhnika*, Minsk 1987. (in Russian)