Computation of the ellipsoidal height related to the Bessel ellipsoid

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Výpočet elipsoidickej výšky nad Besselovym elipsoidom

Príspevok sa zaoberá určením elipsoidických výšok identických bodov nad Besselovym elipsoidom s použitím navrhovanej Molodenského formuly v prípade, že nie je k dispozícii kvázigeoid model vztiahnutý na Besselov elipsoid. Inak, je to redukcia nadmorskej výšky na Besselovom elipsoide a naopak. Ďalej použitím súradníc bodov ŠPS (Štátna priestorová sieť) za účelom určenia transformačných parametrov pomocou MNŠ, určením nadmorskej výšky priamo z GPS meraní a gravimetrickým kvázigeoid modelom Slovenska.

Key words*: Bessel ellipsoidal height, transformation parameters determination, quasi-geoid height above Bessel ellipsoid, weighted least squares computation of datum transformation.*

Introduction

The reference coordinate system in Slovak republic is the S-JTSK (System of Uniform Trigonometric Cadastral Network) where the location and orientation of this network is on the surface of the Bessel ellipsoid (Abelovič, J., et al., 1990). In the absence of distortions in geodetic networks, independent determination of the coordinates of common stations on another datum should yield identical values of the transformation parameters. However, the actual situation is more complicated. Distortions exist in the Slovak geodetic datum due to many reasons such as large-scale and local deformations and the orientation error on the ellipsoid (Abelovič, J, et al., 1990, Priam, Š., 1997, and Kostelecký, J., 1998); add to these reasons, the ambiguity of the S-JTSK coordinates accuracy.

For almost every GPS campaign one has to calculate local transformation parameters in order to minimize the transformation errors. The elevation of points was determined essentially worse than the horizontal position coordinates; the most of S-JTSK network stations have trigonometric heights. If the seven transformation parameters must be determined from identical points, then reduction of the normal heights to their values on the Bessel ellipsoid would be required. For most currently engineering applications, this reduction is not considered and in most cases, the normal or the trigonometric height of the identical points is used instead of the ellipsoidal height above Bessel ellipsoid.

Determination of the transformation parameters

To find the coordinate differences between the datums, some of triangulation points are used to determine seven conversion parameters between the Slovak datum and ETRS89. The seven parameters are three translations or shifts between associated ellipsoid centers ($\Delta X, \Delta Y, \Delta Z$), three rotations angles around *X, Y*, and *Z* axis ($\omega_X, \omega_Y, \omega_Z$) and a scale factor (δS). These parameters can be obtained from a set of points whose positions (coordinates) are known in the both datums. The mathematical model of Molodensky-Badekas will be used in this paper and the seven transformation parameters between the two datums (S-JTSK and ETRS89) will be determined by least squares technique. The procedures to obtain the seven transformation parameters between GPS and S-JTSK are:

Reduction of the S-JTSK identical point's plane coordinates *X* and *Y* to ellipsoidal coordinates φ and λ on the Bessel ellipsoid, equations used to transform the *X, Y* coordinates to the Bessel ellipsoidal coordinates can be found in more details in (Sedlák, V., 2001 and Kuska, F., 1960).

$$\left(X,Y\right)^{\text{S-JTSK}} \rightarrow \left(\varphi,\lambda\right)^{\text{Bessel}} \tag{1}$$

- Reduction of the normal height H to the Bessel ellipsoidal height *h*,

$$\mathrm{H} \rightarrow h \tag{2}$$

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- Conversion of the Bessel ellipsoidal coordinates (φ,λ,h) to the Bessel Cartesian coordinates (X,Y,Z) (Vykutil, J., 1982).

$$(\varphi,\lambda,h)^{\text{Bessel}} \rightarrow (X,Y,Z)^{\text{Bessel}} \quad (3)$$

- Calculation of the coordinates of the center point of the intended area in ETRS89 datum $X_0, Y_0, and\ Z_0$:

$$X_0 = \frac{\sum_{i=1}^{n} X_i}{n}, \quad Y_0 = \frac{\sum_{i=1}^{n} Y_i}{n}, \quad Z_0 = \frac{\sum_{i=1}^{n} Z_i}{n} \quad (4)$$

Where *n* is the number of the identical points and X_i, Y_i *and* Z_i are the ETRS89 geocentric coordinates of identical points.

- Solution by the least squares technique (weighted or un-weighted model) to obtain the most probable values of the transformation parameters.

$$\hat{R} = \left(A^T P A\right)^{-1} A^T P L \quad (5)$$

$$V = A\hat{R} - L \quad (6)$$

$$\sigma_0^2 = V^T P V / (n - u) \quad (7)$$

$$\Sigma_{\hat{R}} = \sigma_0^2 \left(A^T P A\right)^{-1} \quad (8)$$

where $\hat{R}$ is the vector of the adjusted transformation parameters, *V* are the residuals vector of the observations (coordinate differences), σ_0^2 is the a posteriori reference variance, $\Sigma_{\hat{R}}$ is the variance covariance matrix of the adjusted transformation parameters, *u* is the number of unknown transformation parameters and *P* is the weight matrix of the coordinates differences between S-JTSK and ETRS89 datums. The weight matrix is the inverse matrix of the variance covariance matrix of these coordinate differences. *A* is the coefficient matrix.

$$A_{(3n,7)} = \begin{bmatrix} 1 & 0 & 0 & X_1 - X_o & 0 & -(Z_1 - Z_o) & Y_1 - Y_o \\ 0 & 1 & 0 & Y_1 - Y_o & Z_1 - Z_o & 0 & -(X_1 - X_o) \\ 0 & 0 & 1 & Z_1 - Z_o & -(Y_1 - Y_o) & X_1 - X_o & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 1 & 0 & 0 & X_n - X_o & 0 & -(Z_n - Z_o) & Y_n - Y_o \\ 0 & 1 & 0 & Y_n - Y_o & Z_n - Z_o & 0 & -(X_n - X_o) \\ 0 & 0 & 1 & Z_n - Z_o & -(Y_n - Y_o) & X_n - X_o & 0 \end{bmatrix} \quad (9)$$

$$L_{(3n,1)} = \begin{bmatrix} \left(c^E - c^W\right)_1 \\ \vdots \\ \left(c^E - c^W\right)_n \end{bmatrix} \quad (10)$$

Where c^E is the vector of the Bessel Cartesian coordinates of the identical points and c^W is the vector of the geocentric coordinates of the identical points in the ETRS89 datum. More details about this transformation can be found in (Šutti, J, et al., 1997, Sedlák, V., 2001 and Hefty, J., et al., 2003).

The accuracy information of GPS coordinates of identical and new points can be obtained from GPS post processing software and if the accuracy information of S-JTSK coordinates is available, then a weighted least squares computation of the transformation parameters can be performed. The variance covariance matrix of the coordinate differences equals to the sum of the variance covariance matrix of Bessel Cartesian coordinates and the variance covariance matrix of ERTS89 coordinates.

Computation of the Bessel ellipsoidal height

Reduction of the normal height H to ellipsoidal height h is necessary to calculate the Bessel Cartesian coordinates in the case of seven transformation parameters determination. Using these Cartesian coordinates with ETRS89 geocentric coordinates of the identical point makes the determination of the transformation parameters possible.

The normal, ellipsoidal and quasigeoid heights are geometrically related by the following equation (Vykutil, J., 1982). These three heights are illustrated in figure 1

$$h = H + \zeta \tag{11}$$

Where h is the ellipsoidal height related to the Bessel ellipsoid, H is the normal height related to the Baltic Sea after the adjustment and ζ is the quasigeoid height above the Bessel ellipsoid.

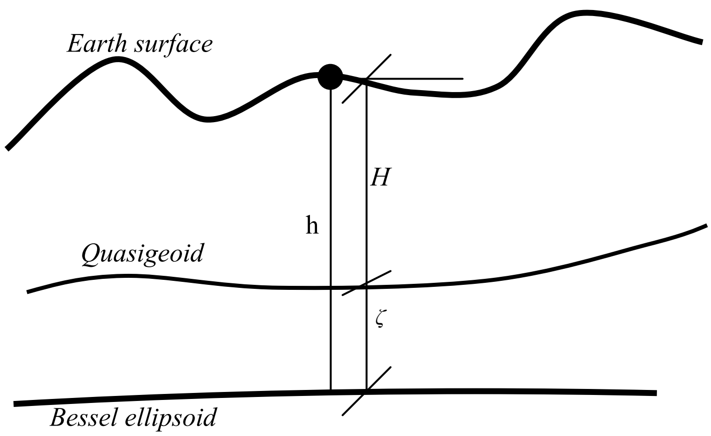

Fig. 1. Definition of the normal, ellipsoidal and quasigeoid heights

The problem of the equation 11 is the value of the quasigeoid height above the Bessel ellipsoid, where can we obtain this value from? How can we obtain the ellipsoidal height related to the Bessel ellipsoid? To obtain the ellipsoidal height above Bessel ellipsoid we have the following situations:

- The precise method to obtain the ellipsoidal height is from the quasigeoid models (Kostelecký, J., 1993). If the quasigeoid model related to the Bessel ellipsoid is available, then using the Bessel ellipsoidal coordinates, the quasigeoid height over Bessel ellipsoid can be obtained and then the ellipsoidal height above Bessel ellipsoid will be calculated from equation 11 where the normal height of the identical points is known.
- If the quasigeoid model is not available and the quasigeoid in the survey area can be approximated to the plane, the normal height can be used instead of the ellipsoidal height. This approximation is applicable if the intended area is not change significantly in the height elements.
- From the following suggested method, the ellipsoidal height above Bessel ellipsoid can be computed using Molodensky Formulas. The standard Molodensky Formula for the height correction is given as (Thomas, A., 1978):

$$\Delta h = \Delta X \cos\varphi_i \cos\lambda_i + \Delta Y \cos\varphi_i \sin\lambda_i + \Delta Z \sin\varphi_i + (a^{Bessel}\ \Delta f + f^{Bessel}\ \Delta a)\sin^2\varphi_i - \Delta a \tag{12}$$

Where Δh is the difference between the Bessel ellipsoidal height and the GPS ellipsoidal height, $\Delta X, \Delta Y,$ *and* ΔZ are the shifts between the ellipsoid centers of the Bessel and GPS ellipsoids, φ_i, λ_i are the ellipsoidal coordinates of identical points related to the Bessel ellipsoid and a, f are the parameters of the Bessel ellipsoid and the differences between their parameters are

$$\begin{aligned} \Delta a &= a^{GPS} - a^{Bessel} \\ \Delta f &= f^{GPS} - f^{Bessel} \end{aligned} \tag{13}$$

The ellipsoidal height over the Bessel ellipsoid of one identical point can be computed from the following equation

$$h^{Bessel} = h^{GPS} - \Delta h \tag{14}$$

Where, h^{GPS} is the ellipsoidal height from GPS survey and Δh is given from Equation 12.
The quasigeoid height of one identical point over the Bessel ellipsoid can be computed as

$$\zeta = h^{Bessel} - H \tag{15}$$

where H is the normal height of the identical point in S-JTSK datum.

The S-JTSK plane coordinates of identical points will be transformed to the Bessel ellipsoidal coordinates and their normal heights will be transformed to the Bessel ellipsoidal heights using Abridged Molodensky Formulas. Then, the average of the height corrections and quasigeoid height of the all given identical points can be computed as follows

$$\Delta h_{avg} = \frac{\sum_{i=1}^{n} \Delta h_i}{n} \qquad (16)$$

$$\zeta_{avg} = \frac{\sum_{i=1}^{n} \zeta_i}{n} \qquad (17)$$

where n is the number of identical points.

Assuming that, we have four identical points, then for each identical point, the height correction can be computed from equation 12 and the average of the four height corrections can be computed from equation 16. Using the normal heights of the identical points, then their quasigeoid heights can be computed from equation 15 and then the average of them can be obtained from Equation 17. The quasigeoid height of each unknown point j above the Bessel ellipsoid can be obtained from the following equation:

$$\zeta_j = \frac{\Delta h_j \times \zeta_{avg}}{\Delta h_{avg}} \qquad (18)$$

Where Δh_j is the height correction for each unknown point j which can be obtained from equation 12 using the transformed ellipsoidal coordinates (Bessel ellipsoid) of each unknown point resulted from the transformation and the ellipsoidal height above GPS ellipsoid.

The parameters values used for calculation are shown in the Table 1. The suggested values of the translations between S-JTSK and GPS datums shown in the table 1 were obtained using 8 identical points entire Slovakia (Mojzeš, M., 2004). If the height information of identical points and quasigeoid model related to the Bessel ellipsoid are available or not, the previously method can be used.

Parameter	Value
ΔX	579.04 m
ΔY	67.22 m
ΔZ	485.80 m
Δa	739.845 m
Δf	0.000010037499608
a^{Bessel}	6377397.155 m
f^{Bessel}	0.003342773181578

Tab. 1. Parameters used to calculate the Bessel ellipsoidal height

Numerical Test

Six pints from the ŠPS network (Vyšná Myšla, Cecejovce, Nižná Kamenica, Paňovce, Vyšný Klátov and Šarišské Bohdanovce) with their coordinates in both datums (ETRS89 and S-JTSK) were chosen as identical points to determine the transformation parameters between the two datums and then to obtain the S-JTSK coordinates of two new points (Košice 2 and Košice 3). The following procedures were performed:

- The first computation of transformation parameters was performed with neglecting of the quasigeoid height above the Bessel ellipsoid (the normal height equals to the Bessel ellipsoidal height).
- The Bessel ellipsoidal height of the identical points was computed from Molodensky formula (equation 12) and the normal height of the new points was computed using equations (16, 17, 18 and then 11).

The maximum residual of the first case was 15 cm with the average of 4 cm and the maximum difference between the original and the transformed S-JTSK coordinates of identical points is 19 cm with the average of 4 cm, and the maximum residual and coordinate differences for the second case is 5 and 1 cm respectively.

The accuracy of the transformation parameters and the accuracy of the transformed new points which obtained in the case of using the Molodensky Formula to compute the Bessel ellipsoidal heights of identical points are better than in the case of using the normal height as the ellipsoidal height. Rotations angles and the scale factor in the second case (Molodensky) are also smaller than the first case.

Unfortunately, the heights of the identical and new points are trigonometric heights. The maximum trigonometric height of the identical points in the survey area was about 467 m with the minimum of 225 m.

The standard deviations of the trigonometric heights were assumed to be 6 cm and the standard deviations of S-JTSK horizontal coordinates were between 2 and 3 cm (the same accuracy was used for the two cases)

The results are shown in the following Tables:

Tab. 2. Datum Transformation Parameters in two cases

Transformation parameters	**With neglecting quasigeoid height above Bessel**		**With considering quasigeoid height above Bessel**		**Unit**
ΔX	-582.615	0.025	-579.277	0.016	m
ΔY	-72.886	0.014	-71.589	0.009	m
ΔZ	-488.356	0.029	-484.279	0.019	m
δS	-7.696	0.759	-6.803	0.511	ppm
ω_X	3.548	0.679	5.533	0.457	sec
ω_Y	1.696	0.819	2.233	0.551	sec
ω_Z	8.787	0.459	6.995	0.309	sec

Tab. 3. S-JTSK Coordinates of new points in the case of the normal and Bessel ellipsoidal height are equivalent

station	**Without Jung's Transformation**	**With Jung's Transformation**
	S-JTSK transformed coordinates [m]	**S-JTSK transformed coordinates** [m]
KE2	265484.75±0.010	265484.75±0.010
	1239504.61±0.013	1239504.61±0.013
	302.355±0.037	302.287±0.037
KE3	258567.98±0.010	258567.99±0.010
	1238566.27±0.013	1238566.28±0.013
	307.905±0.037	307.896±0.037

Tab. 4. S-JTSK Coordinates of new points (the Bessel ellipsoidal height based on Molodensky Formula)

station	**Without Jung's Transformation**	**With Jung's Transformation**
	S-JTSK transformed coordinates [m]	**S-JTSK transformed coordinates** [m]
KE2	265484.75±0.007	265484.75±0.007
	1239504.61±0.008	1239504.61±0.008
	302.363±0.025	302.363±0.025
KE3	258567.98±0.007	258567.99±0.007
	1238566.27±0.008	1238566.28±0.008
	307.850±0.025	307.849±0.025

Table 5 contains the heights of all points using the new gravimetric quasigeoid model (GMSQ98BF). The second column in the Table 5 contains the published ellipsoidal heights related to the (ETRS89 datum) and the third column contains the quasigeoid heights obtained from the gravimetric quasigeoid model (GMSQ98BF). The forth column contains the quasigeoid heights obtained by subtracting the GRS80 ellipsoidal heights (second column) from known trigonometric heights (sixth column). The fifth column contains the trigonometric heights (second column minus third column) and the final column is the difference between the known heights and the heights obtained from the GPS and gravimetric model. In this example, the differences between the known heights (original) and (transformed heights obtained from Tab. 3 and Tab. 4) for the Košice 2 and Košice 3 stations are greater than the differences obtained from Table 5 (column 7).

Tab. 5. the heights of all points obtained from the quasigeoid model (GMSQ98BF), all values are in meters

station	*h* from GPS	ζ^{grav}	ζ^{GPS}	H^{GPS}	Given H	$H - H^{GPS}$
Vyšná Myšla	265.602	40.033	40.012	225.569	225.590	0.021
Cecejovce	333.256	40.800	40.786	292.456	292.470	0.014
Nižná Kamenica	382.160	39.729	39.710	342.431	342.450	0.019
Paňovce	358.893	40.809	40.803	318.084	318.090	0.006
Vyšný Klátov	508.788	40.790	40.828	467.998	467.960	-0.038
Šarišské	284.928	39.939	39.968	244.989	244.960	-0.029
Košice 2	342.733	40.404	40.423	302.329	302.310	-0.019
Košice 3	348.028	40.057	39.938	307.971	308.090	0.119

Summary and Conclusions

If the strategy to densifying of S-JTSK using GPS technique and then transformed the coordinates of new points to S-JTSK plane using local transformation parameters, the following notes can be considered:

1. Some points of the State spatial network (ŠPS) have known coordinates in S-JTSK datum and all of these points have known coordinates in ETRS89 datum, thus, each GPS survey can be connected to at least three stations from (ŠPS) to obtain all other new points in ETRS89 datum and to use them as identical points for transformation purposes. More identical points are preferable; in this case, other identical points with a good geometrical configuration surrounding the area of densification can be used without occupying them by GPS receivers.
2. It's better if those identical points have normal heights from levelling and if the accuracy of the height element is not important, the gravimetric quasigeoid model can be used to determine the normal height directly from GPS survey.
3. The ellipsoidal height of the identical points above Bessel ellipsoid can be computed from the Molodensky formula especially if there is no height information and quasigeoid model available.

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