

Ultracapacitor Modelling and Control Using Discrete Fractional Order State-Space Model

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Ultrakapacitné modelovanie a riadenie použitím diskretného čiastočného riadenia modelu s roztiahnutými parametrami.

In this paper the modelling of ultracapacitor system using discrete fractional order state-space system is presented. The obtained model is used for design and testing of state feedback controller with observer.

Key words: *Ultracapacitor Modelling, State-Space Model, Discrete Fractional Order*

Introduction

Ultracapacitors (aka supercapacitors) are electrical energy storage devices which offer high power density which was not possible to achieve in traditional capacitors. There are many approaches to ultracapacitors modelling. Many of the authors use more or less complicated RC models (Buller, Karden, Kok, Doncker, 2002), which are accurate especially for low frequencies. Some of the authors describe ultracapacitors by RC transmission line (Belhachemi, Rael, Davat, 2000). In the papers (Quintana, Ramos, Nuez, 2006; Westerlund, Ekstam; 1994) very efficient approach using fractional order calculus was presented. In this paper the fractional order calculus approach to ultracapacitor modelling is used, especially discrete fractional order state-space model (DFOSS).

Foundations of ultracapacitors technology

The capacity of typical ultracapacitors is many times bigger than typical capacitors and has values from fractional parts of Farad to thousands of Farads. Its energy density has values about 2...9 Wh.kg⁻¹ and power density is 3...8kW.kg⁻¹. These values locate ultracapacitors between electrolytic capacitors and batteries.

Thanks to their huge capacity, and small dimensions the ultracapacitors have found numerous applications in modern technology.

One of the most prominent is the ultracapacitor application in hybrid cars (Burke, 2001; Wight, 2002). When a hybrid car is decelerating the electric motor acts as a generator producing a short, but high value energy impulse. This is used to charge the ultracapacitor. Charging the conventional batteries with such a short impulse would be extremely ineffective. Similarly, during start-up of the electric motor a short-time but substantial in value increase of the source power is needed. This is achieved by using the ultracapacitor.

Other areas of ultracapacitor applications is in power electronics converters (mainly inverters) with DC circuit (Wodecki, Koczara, 2004; Rufer, Barrade, 2002).

Ultracapacitors can be found in wind power stations (Abbey, Joos, 2007), where they stabilise the power supplied to the grid. They are charged during the period of strong wind and discharge during calm periods. They can also be applied as energy saving subsystems in underground energy supply system. They are placed along the tracks and they collect the energy during braking and give it back during start-up. Also some back-up systems in electronics and IT use ultra capacitors (e.g. computer memory back-up).

In most of the applications mentioned it is essential to have a fairly detailed model of ultracapacitor. This model makes the design of control systems possible. The more accurate model we have, the more advanced control schema can be achieved. Control systems are needed e.g. to stabilise the ultracapacitor voltage which tends to fluctuate significantly.

Ultracapacitor is build from two activated carbon plates mixed with electrolyte and separator (Zorpette, 2005). Its working rule is based on Helmholtz effect, who found out that there exists some level of voltage below which the electrolysis does not take place. Below this level the electrolyte behaves as an insulator. When to the ultracapacitor electrodes the voltage below this level is connected the electrolysis does not start and the current does not flow through the electrolyte. However, in this case, the motion of ions contained in electrolyte (positive to negative electrode and negative to positive electrode) takes place. Because

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of the absence of electrolysis, at the surface between electrode and electrolyte a very thin isolation film is formed. Thanks to this and to the very large area of the surface of the electrodes made of porous activated carbon (500-2000 m²/g) it is possible to achieve such a large capacity.

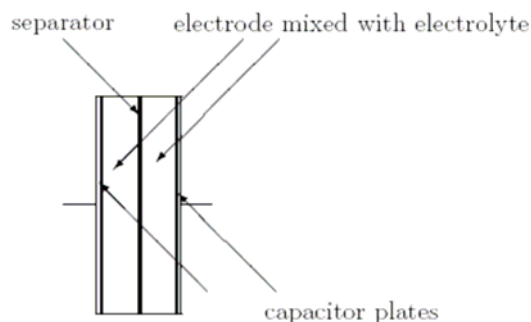


Fig. 1. Ultracapacitor general idea.

Fig. 1 summarises this general idea of the ultracapacitor. The ultracapacitor electrodes are usually made of activated carbon soaked with electrolyte (e.g. sulphuric acid solution H₂SO₄) separated with thin porous membrane. This separator prevents the electrodes from short circuiting. It allows however, the movement of ions in the electrolyte.

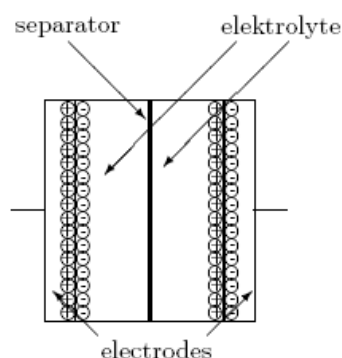


Fig. 2. Ultracapacitor charging.

Fig. 2 explains the process of ultracapacitor charging. The voltage applied to the plates of the ultracapacitor causes the movement of the ions. Negative ions move to the positive electrode (anode), and positive ones move to the negative (cathode). At the border between the electrodes and the electrolyte two layers are formed. These layers collect electrons and positive ions. This is why the ultracapacitors are sometimes called double-layer capacitors (Zorpette, 2005). The detailed description of the electrochemical processes going on in this area is found in (Endo, Takeda, Kim, Koshiba, Ishii, 2001).

Fractional order calculus

The fractional calculus (generalization of a traditional integer order integral and differential calculus) idea has been mentioned in 1695 by Leibnitz and L'Hospital. In the end of 19th century Liouville and Riemann introduced first definitions of fractional derivative.

Continuous time fractional order calculus

There exist two main definitions of the fractional order integrals and derivative: Riemann-Liouville and Grünwald-Letnikov. Both of those definitions are equivalent and for $\alpha > 0$ give results of the fractional order derivative, for $\alpha < 0$ fractional order integral and for $\alpha = 0$ the same function. This is why those definitions are called differ-integrals definitions. The Riemann-Liouville definition was derived from the Cauchy iterated integral formula and is given as follows:

Definition 1 (Riemann-Liouville) The following operator:

$${}_a D_x^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int_a^x (x-u)^{-\alpha-1} f(u) du & \alpha < 0 \\ \frac{d^m}{dx^m} \left[\frac{1}{\Gamma(m-\alpha)} \int_a^x (x-u)^{m-\alpha-1} f(u) du \right] & \alpha > 0 \end{cases}$$

is called fractional order differ-integral.

where

$$m - 1 \leq \alpha < m$$

and $\alpha \in \mathbb{R}$ (\mathbb{R} – set of real numbers) is a fractional order of the differ-integral of the function $f(x)$.

The Grünwald-Letnikov definition was derived from the iterated derivative formula and is given as follows:

Definition 2 (Grünwald-Letnikov) The following operator

$$D^\alpha x(t) = \lim_{h \rightarrow 0} \frac{1}{h^\alpha} \sum_{j=0}^k (-1)^j \binom{\alpha}{j} x(t - jh) \tag{1}$$

is called fractional order differ-integral.

Where $\alpha \in \mathbb{R}$ is an order of the fractional differ-integral and $k = [\frac{t}{h}]$. The factor $\binom{n}{j}$ can be obtained from relation:

$$\binom{n}{j} = \begin{cases} 1 & \text{for } j = 0 \\ \frac{n(n-1)\dots(n-j+1)}{j!} & \text{for } j > 0 \end{cases} \tag{2}$$

The Laplace transform of the fractional order differ-integral is given as follows:

$$L[{}_0 D_x^\alpha f(x)] = \begin{cases} s^\alpha F(s) & n < 0 \\ s^\alpha F(s) - \sum_{k=0}^{j-1} s^k {}_0 D_x^{\alpha-k-1} f(0) & n > 0, \quad j-1 < \alpha \leq j \in \mathbb{N} \end{cases}$$

Bode diagram of the fractional order itegrator

Let us assume the following transfer function of fractional order integrator

$$G(s) = \frac{1}{Ts^\alpha} \tag{3}$$

then the spectral transfer function is given by

$$G(j\omega) = \frac{1}{T(j\omega)^\alpha} = \frac{1}{T\omega^\alpha (\cos \frac{\pi}{2} \alpha + j \sin \frac{\pi}{2} \alpha)} \tag{4}$$

The magnitude of the transfer function is given as follows (Jifeng & Yuankai 2005)

$$A(\omega) = \sqrt{\frac{\cos^2 \frac{\pi}{2} \alpha + \sin^2 \frac{\pi}{2} \alpha}{T\omega^{2\alpha}}} = \frac{1}{T\omega^\alpha} \tag{5}$$

which gives the logarithmic magnitude in the form:

$$M(\omega) = -20 \log(T) - \alpha 20 \log(\omega) \tag{6}$$

The phase properties are given as follows:

$$\varphi(\omega) = \arg \left[\frac{1}{T\omega^\alpha} j^{-\alpha} \right] = -\alpha \frac{\pi}{2}$$

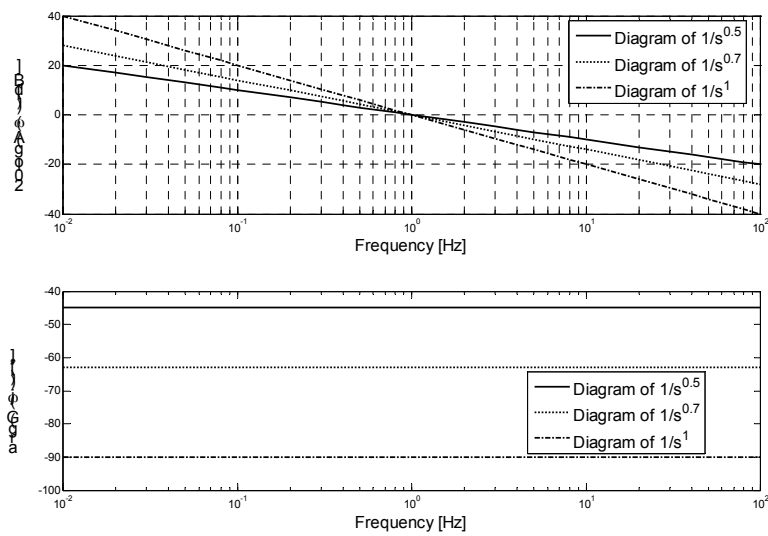


Fig. 3. Bode diagrams of $\frac{1}{s^\alpha}$ systems for $\alpha = 0.5, 0.7, 1$ $T=1$.

Bode diagram of the proportional system with integrator

Let us assume the following transfer function, which is used later for modelling of the ultracapacitor:

$$G(s) = 1 + \frac{1}{Ts^\alpha}$$

the spectral transfer function has the form

$$\begin{aligned} G(j\omega) &= 1 + \frac{1}{T(j\omega)^\alpha} = \frac{T\omega^\alpha (\cos \frac{\pi}{2}\alpha + j\sin \frac{\pi}{2}\alpha) + 1}{T\omega^\alpha (\cos \frac{\pi}{2}\alpha + j\sin \frac{\pi}{2}\alpha)} \\ &= \frac{T\omega^\alpha + \cos \frac{\pi}{2}\alpha - j\sin \frac{\pi}{2}\alpha}{T\omega^\alpha} \end{aligned}$$

The magnitude of the transfer function is given as follows (Jifeng & Yuankai 2005)

$$A(\omega) = \frac{\sqrt{T^2\omega^{2\alpha} + 2T\omega\cos \frac{\pi}{2}\alpha + 1}}{T\omega^\alpha} = \sqrt{1 + \frac{2\cos \frac{\pi}{2}\alpha}{T\omega^\alpha} + \frac{1}{T^2\omega^{2\alpha}}}$$

which yields

$$A(\omega) = \begin{cases} 1 & \omega \gg \frac{1}{T} \\ \frac{1}{T\omega^\alpha} & \omega \ll \frac{1}{T} \end{cases}$$

and finally

$$M(\omega) = \begin{cases} 0 & \omega \gg \frac{1}{T} \\ -20\log(T) - \alpha 20\log(\omega) & \omega \ll \frac{1}{T} \end{cases}$$

The phase properties are given as follows:

$$\varphi(\omega) = \arctg \left[-\frac{T\omega^\alpha \cos \frac{\pi}{2}\alpha + 1}{T\omega^\alpha \sin \frac{\pi}{2}\alpha} \right]$$

Fig. 4 presents example of Bode diagrams for $T = 1$ and $\alpha = 0.5, 0.7, 1$.

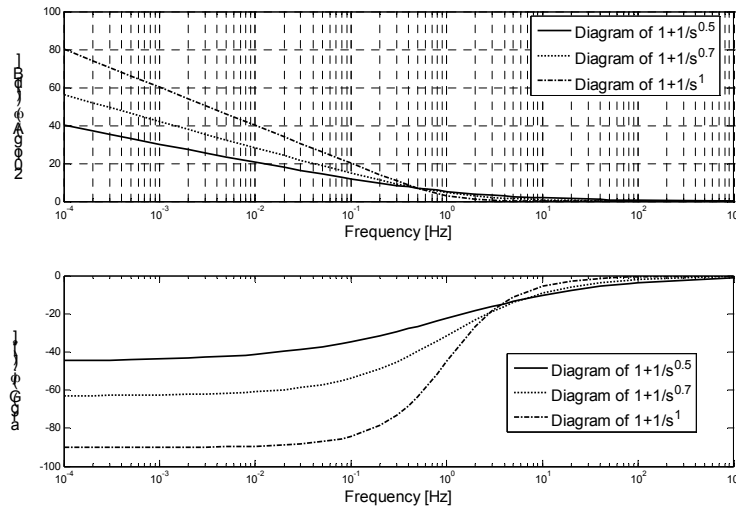


Fig. 4. Bode diagrams of $1 + \frac{1}{s^\alpha}$ systems for $\alpha = 0.5, 0.7, 1$.

Discrete time fractional order calculus

In this paper the following definition of fractional order difference (Oldham, Spanier 1974; Podlubny, 1999) will be used.

Definition 3 The following operator

$$\Delta^n x_k = \sum_{j=0}^k (-1)^j \binom{n}{j} x_{k-j} \tag{7}$$

is called the fractional order difference.

Where $n \in \mathbb{R}$ is an order of the fractional difference, $k \in \mathbb{N}$ (\mathbb{N} - set of natural numbers) is a number of the sample for which the derivative is calculated.

According to this definition it is possible to obtain the discrete equivalent of derivative (when n is positive), the discrete equivalent of integration (when n is negative) or when n equal to 0 the original function.

More properties of the definition are to be found in (Ostalczyk, 2000; 2004a; 2004b; Jun 2001).

Definition 4 The following set of equations

$$\Delta^n x_{k+1} = A_d x_k + B u_k \tag{8}$$

$$x_{k+1} = \Delta^n x_{k+1} - \sum_{j=1}^{k+1} (-1)^j \binom{n}{j} x_{k-j+1} \tag{9}$$

$$y_k = C x_k \tag{10}$$

are called the linear discrete fractional order system in state-space representation, where $n \in \mathbb{R}$ is the order of the system.

The value of fractional order difference of state vector for time instant $k+1$ is obtained according to (8); from this value the state vector x_{k+1} is calculated using relation (9). The output equation is given by (10). More properties of DFOSS are presented in (Dzielinski, Sierociuk, 2006b; 2006a; Sierociuk, Dzielinski, 2006; 2005)

Stability

The article (Dzielinski, Sierociuk, 2006b) presents a stability criterion for the FOSS. It is easy to check and to use in the design of stable control systems. Even though this criterion does not define the whole stability area it is very useful in observer design.

Theorem 5 The system given by the Definition 4 is stable if (but not iff)

$$|A_d + \binom{n}{1} I| < r(k, n)$$

where

$$r(k, n) = \begin{cases} 2 - \frac{\Gamma(k+2-n)}{\Gamma(1-n)\Gamma(k+2)} - n & ; n \in (2,1) \\ \frac{\Gamma(k+2-n)}{\Gamma(1-n)\Gamma(k+2)} + n & ; n \in (1,0) \\ 1 & ; n = 0 \end{cases}$$

Proof is given in (Dzielinski & Sierociuk 2006b).

The $r(n, k)$ is a stability radius of the system, i.e., it is a radius of the disk in which stable eigenvalues (but not all stable eigenvalues) of the system are placed.

Observer

Theorem 6 The state observer for Discrete Fractional State-Space Systems (called Discrete Fractional Order Observer (DFOO)) is given by the following equations

$$\begin{aligned} \Delta^n \hat{x}_{k+1} &= F_d \hat{x}_k + Gu_k + Hy_k \\ \hat{x}_{k+1} &= \Delta^n \hat{x}_{k+1} - \sum_{j=1}^{k+1} (-1)^j \binom{n}{j} \hat{x}_{k-j+1} \end{aligned}$$

where \hat{x}_k is an estimate of the state variable x_k , $F_d = A_d - HC$ and $G = B$.

Proof is given in (Dzielinski, Sierociuk, 2006a).

Ultracapacitor Continuous Time Modelling

In this section frequency domain identification results and methodology for ultracapacitor model are presented. This results is used for identification of the structure of the model (order of the model), this structure will be used in next section for discrete time modelling.

Experimental setup of ultracapacitor system

The experimental setup contains the electronic circuit with ultracapacitor connected to the DS1104 Control Card. The electronic circuit is presented in Fig. 5 and contain operational amplifier OPA544, resistor 180Ω and ultracapacitor $0.22F$. OPA 544 is a high current operational amplifier and works in voltage follower configuration. This circuit is used for both continuous and discrete modelling.

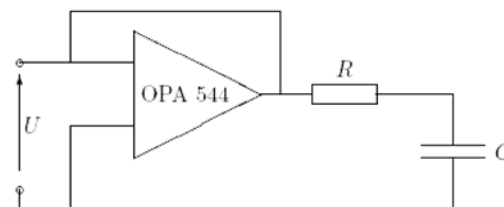


Fig. 5. Electronic circuit of ultracapacitor system scheme.

Ultracapacitor Frequency Modelling

Spectral transfer function is defined as

$$G_{uc}(j\omega) = \frac{U_{uc}(j\omega)}{I(j\omega)}$$

Where $U_{uc}(j\omega)$ is the Fourier transform of capacitor voltage and $I(j\omega)$ is the Fourier transform of the capacitor current. Because the ultracapacitor is an electrolytic capacitor it can only accept positive voltages. That is why a signal with constant component $u(t) = 2 + \sin(\omega t)$ as an input signal was used. Capacitor voltage (in steady state) in this case is equal to $u_c(t) = 2 + A_c(\omega)\sin(\omega t + \varphi_u)$, and capacitor current is $i(t) = A_i(\omega)\sin(\omega t + \varphi_c)$.

The Bode diagram was obtained from the following relations:

$$M(\omega) = 20 \log \left(\frac{A_c(\omega)}{A_i(\omega)} \right), \quad \varphi(\omega) = \varphi_i(\omega) - \varphi_u(\omega)$$

As a theoretical model the following transfer function was used:

$$G_{uc}(s) = K \left(1 + \frac{1}{T_1 s^\alpha} \right)$$

In parameters identification by diagrams matching the following parameters were achieved $K = 30.6277$, $T_1 = 1.5719$, $\alpha = 0.45$

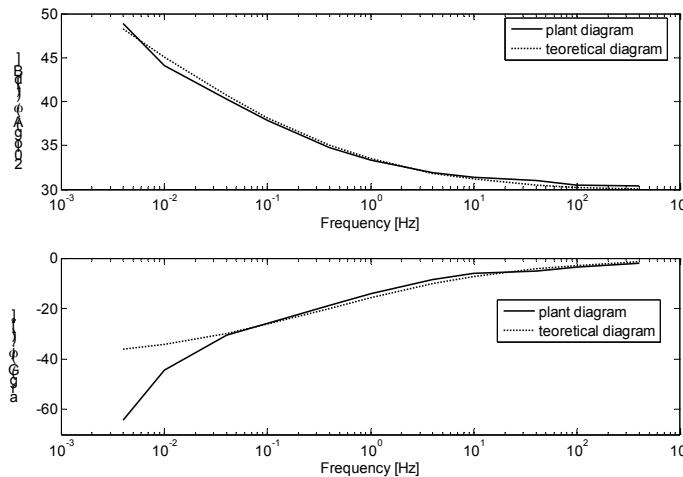


Fig. 6. Measured and theoretical Bode diagrams of ultracapacitor.

The comparison of the measured data and Bode diagram of modelled transfer function is presented in Fig. 6. As it can be seen for frequencies higher than 0,04 Hz the modelling is very accurate. For lower frequencies more complicated model should be used, as it was presented in (Quintana et al., 2006).

Spectral transfer function of the system with ultracapacitor is defined as:

$$G(j\omega) = \frac{U_{uc}(j\omega)}{U(j\omega)}$$

where $U_{uc}(j\omega)$ is the Fourier transform of the capacitor voltage and $U(j\omega)$ is the Fourier transform of the input voltage.

As a theoretical model the following transfer function was used:

$$G(s) = \frac{G_{uc}(s)}{1 + G_{uc}(s)}$$

In parameters identification by diagrams matching the following parameters were achieved $K = 30.2319$, $T_1 = 1.7229$, $\alpha = 0.45$

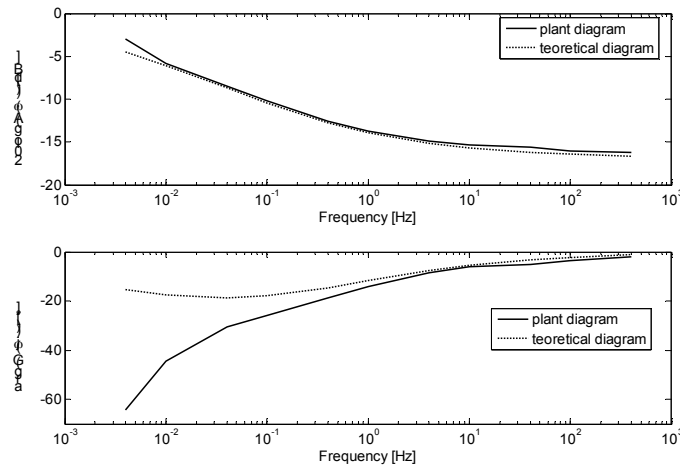


Fig. 7. Measured and theoretical Bode diagrams of system with ultracapacitor.

The comparison of the measured data and transfer function diagram is presented in Fig. 7. And again for lower frequencies more complicated model should be used, as it was presented in (Quintana et al. 2006), but for our work this simple model is accurate enough.

Ultracapacitor discrete time modelling

The continuous fractional order state-space system of the system with ultracapacitor identified by diagrams matching in previous section is given as follows:

$$\frac{d^\alpha x}{dt^\alpha} = [a_{0,0}]x(t) + [b_0]u(t)$$

$$y = [c_0]x(t) + [d_0]u_k$$

The modelling of such a system is not easy because of direct connection between input and output system by matrix D (d_0). In a control system this may lead to the algebraic loop. The value of resistance R was chosen so high in order to minimize this effect, but it still exists. That is why for discrete modelling the system with two state variables and order equal to $\alpha=0.2$ was chosen.

By output error minimalization the following discrete fractional order state-space system is obtained:

$$A_d = \begin{bmatrix} 0 & 1 \\ 0.035311 & 0.001815 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \alpha = 0.2$$

$$C = [-0.018624 \quad 0.188432] D = [0], T_s = 0.1s$$

Results of discrete fractional order modelling of the system with ultracapacitor are presented in Fig. 8. Because of unknown initial state vector values of the system real input signal contains 0,2 V constant component which is not presented in the figures.

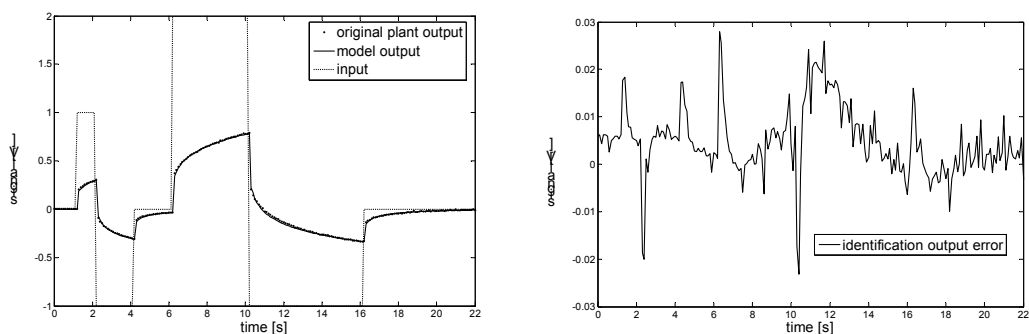


Fig. 8. Results of discrete modelling of system with ultracapacitor.

Estimation and control experiment

The observer for the discrete fractional order state space systems and design method was introduced in (Dzielinski & Sierociuk 2006a).

Eigenvalues of the model obtained in previous section are equal to:

$$\text{eig}[A_d + \text{diag}(N)] = [0.013 \quad 0.389]$$

Stability radius given by stability criterion presented in (Dzielinski & Sierociuk 2006b) for $L=\infty$ is equal to 0.2. Eigenvalues of the observer system matrices are chosen in order to guarantee asymptotic stability of estimation error equation and to make the observer faster than the observed system.

$$\text{eig}[F + \text{diag}(N)] = [0.117 \quad 0.0039]$$

For chosen eigenvalues the following observer is obtained, according to method presented in (Dzielinski, Sierociuk, 2006a).

$$F = \begin{bmatrix} 0.30530170627342 & -2.08898904996778 \\ 0.09328584555151 & -0.58475725385212 \end{bmatrix}, G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$H = \begin{bmatrix} 16.39311378327184 \\ 3.11290941602129 \end{bmatrix}$$

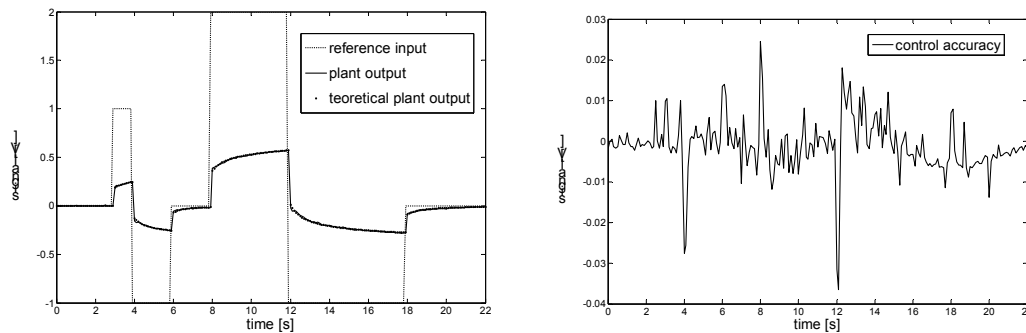


Fig. 9. State feedback control results.

When estimated state variables are accessible the state feedback control can be applied. Let us assume the regulator matrix as follows:

$$K = [0.05 \quad 0.05]$$

Fig. 9 presents results of state feedback control of the system with ultracapacitor. The left figure shows the system output compared with a model given by system matrix $A - BK$. The right figure presents the system input and reference input. These results are presented without $2V$ constant component.

Conclusions

In the paper the results of modeling and control of the ultracapacitor system were presented. The ultracapacitor is an inherently fractional order dynamics. Therefore the discrete fractional order state-space system introduced by the Authors seems to be an excellent alternative to the other approaches of modeling and control.

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