Iterative Optimal Control for Flexible Rotor–AMB System

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In this paper we propose an optimal control design technique for a class of nonlinear and control non-affine equations. The dynamic equations of a flexible shaft supported by a pair of active magnetic bearings (AMBs) are used as the nonlinear control non-affine equations. The basic model of the system will be a Timoshenko beam supported at the ends by magnetic frictionless bearings. The effective control of such systems is extremely important for very high angular velocity shafts which are a feature of many modern machines. The control must be able to cope with unbalanced masses and hence be very robust.

In this paper we shall approach the problem by discretising the Timoshenko beam model and using standard difference formulae to develop a finite-dimensional model of the system. Then we use a recently developed technique for controlling nonlinear systems by reducing the problem to a sequence of linear time-varying (LTV) systems. An optimal control designed for each approximating linear, time-varying system and recent results show that this method will converge uniformly on compact time intervals to the optimal solution.

Key words: *Approximation scheme, LQR control, Timoshenko beam, Active Magnetic Bearings.1 Introduction*

Introduction

In the rotordynamics applications elastic behaviour of a rotating shaft is an important concern. There are several works in the literature on the modelling and vibration control of a flexible shaft (Das et.al., 2008). Since the analysis of flexible structure with lumped parameters does not reflects the real behaviour, continuous time models with partial differential equations are used for more realistic analysis.

The well known classical beam theory also called as Euler-Bernoulli beam theory has been used by many authors for the vibration analysis of the shafts. However, this beam theory is not suitable for thick beams since the shear deformation and rotary inertia effects are not included in the model.

Timoshenko's theory of beams constitutes an improvement over the Euler-Bernoulli theory, in that it incorporates shear and rotational inertia effects (Graph, 1975). By also taking gyroscopic effects into account Timoshenko beam equation is given as

$$\begin{aligned} &E\,I\frac{\partial^4 y}{\partial x^4}-\left(\rho\,I+\frac{E\,I\,\rho}{\kappa\,G}\right)\frac{\partial^4 y}{\partial x^2\,\partial t^2}+\rho\,A\,\frac{\partial^2 y}{\partial t^2}+\\ &+\frac{\rho^2\,I}{\kappa\,G}\frac{\partial^4 y}{\partial t^4}-2\rho\,I\,\Omega\frac{\partial^3 z}{\partial x^2\,\partial t}+\frac{2\rho^2 I\,\Omega}{\kappa\,G}\frac{\partial^3 z}{\partial t^3}\\ &=f(x,y,t). \end{aligned} \tag{1}$$

Here *E, G, I, ρ, Ω* and *κ* denotes respectively modulus of elasticity, shear modulus of elasticity, area moment of inertia of the shaft cross section, mass density, shaft rotation speed and Timoshenko shear coefficient. This shear coefficient is a dimensionless and shape dependent parameter and for a circular cross section it is calculated from Eq.(2) where *v* is the poisson's ratio for shaft material.

$$\kappa=\frac{6\left(1+\nu\right)}{\left(7+6\nu\right)} \tag{2}$$

Last term *f(x,y,t)* in the Eq.(1) is given as the distributed force term. In our case shaft is supported by two AMBs at both ends and therefore *f(x,y,t)* will be magnetic levitation force effecting only at the boundaries of the shaft.

AMBs are used to levitate the rotating body in the air electromagnetically and hence there will be no contact between the parts and provide adjustable stiffness and damping ratios. However, AMBs are inherently unstable and require feedback control to keep the shaft at the centre position. (Scheweitzer, 1976) derived magnetic bearing forces using linear magnetic circuit theory and produced nonlinear force equation is given as

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$$f = \frac{1}{4}\mu_0 \; N^2 A \cos\alpha \frac{i^2}{s^2} \tag{3}$$

where μ_0 is the relative permeability of the air and is equal to $4\pi \times 10^{-7}$ N/A^2, N is the number of coils around the stator block, A is the cross-sectional area of the electromagnet, α is half of the angle between stator poles, i is the control current and s is the distance of the shaft's centre to the bearing centre.

If we assume there are two electromagnets on the opposite sides of the ferromagnetic rotor and working in the so-called differential driving mode then the Eq.(3) for an AMB in one direction will be (Tombul et.al., 2009)

$$f_y = k\left[\frac{(i_b - i_y)^2}{(s+y)^2} - \frac{(i_b + i_y)^2}{(s-y)^2}\right] \tag{4}$$

The parameter i_b is called the bias current and supplied to each electromagnet constantly. Since the electromagnetic actuators in the opposite directions work in the differential driving mode i_y current is added to one and subtracted from the other in order to produce a net force in the required side.

Our aim in this paper is to find the optimal value of the i_y current by solving the differential Riccati equation which is subjected to minimise a performance index given in Eq. (5) and hence to control the deviation in the shaft.

$$\begin{aligned} J &= \frac{1}{2} x^T(t_f)\; S\; x(t_f) \\ &+ \frac{1}{2}\int_{t_0}^{t_f} \{x^T(t)\; Q\; x(t) + u^T(t)\; R\; u(t)\} dt \end{aligned} \tag{5}$$

Timoshenko beam equation is derived from linear elasticity theory so it is purely a linear model. However, the magnetic forces are highly nonlinear and resulting mathematical model in state space will be nonlinear and non-affine due to these force terms.

Provided the LQR problem is only applicable for linear systems we are suggesting in this paper simplifying the nonlinear system as a sequence of LTV systems and iterating them in a finite-time interval. Nonlinear dynamics of flexible shaft-AMB system is used for the numerical simulations. It is shown that after a few iterations response of the approximated LTV systems converges uniformly to the nonlinear system's response.

LQR Nonlinear control

Consider the optimal control of such non-affine nonlinear dynamics of a system which can be expressed in the state space form as,

$$\begin{aligned} &\dot{x}(t) = A\,(x(t))\, x(t) + B\,(x(t), u(t))\, u(t)\,, \\ &x(t_0) = x_0\,, \end{aligned} \tag{6}$$

subjected to minimise the quadratic cost functional given by Eq.(5). To design an LQR control for the nonlinear system the following sequence of approximations can be introduced (Banks and McCaffrey, 1998),

$$\begin{aligned} \dot{x}^{[i]}(t) &= A\left(x^{[i-1]}(t)\right) x^{[i]}(t) \\ &+ B\left(x^{[i-1]}(t), u^{[i-1]}(t)\right) u^{[i]}(t) \end{aligned} \tag{7}$$

where $x^{[i]}(t_0) = x_0$ and then the corresponding linear quadratic cost functional for this approximated LTV system becomes,

$$\begin{aligned} J &= \frac{1}{2} x^{[i]T}(t_f)\; S\; x^{[i]}(t_f) \\ &+ \frac{1}{2}\int_{t_0}^{t_f} \left\{x^{[i]^T} Q x^{[i]} + u^{[i]^T} R u^{[i]}\right\} dt \end{aligned} \tag{8}$$

Here in Eq.(8) $S, Q \in \Re^{n\times n}$ are both symmetric and positive semi definite matrices whereas $R \in \Re^{m\times m}$ is symmetric but can only be positive definite. Since the only known value of the state is its initial condition, for the first approximation $x^{[i-1]}(t)$ has been assumed to be x_0. Also another assumption for the control in the first approximation is made and control is chosen as $u^{[i-1]}(t) = 0$.

Apart from the first sequence of Eqs. (7) and (8) each approximating problem is now linear time varying (LTV). Therefore, classical finite time LQR optimal control theory can be applied to this sequence, which results in the linear control law.

$$u^{[i]}(t) = -R^{-1}B^{T}\left(x^{[i-1]}(t),u^{[i-1]}(t)\right)P^{[i]}(t)x^{[i]}(t) \tag{9}$$

Positive definite and time-varying $P \in \Re^{n\times n}$ is the solution of differential Riccati equation which is also given as a sequence of linear time-varying systems as follows,

$$\begin{aligned} \dot{P}^{[i]}(t) &= -Q - P^{[i]}(t)A\left(x^{[i-1]}\right) - A^{T}\left(x^{[i-1]}\right)P^{[i]}(t) \\ &+ P^{[i]}(t)B\left(x^{[i-1]},u^{[i-1]}\right)R^{-1}B^{T}\left(x^{[i-1]},u^{[i-1]}\right)P^{[i]}(t) \end{aligned} \tag{10}$$

Since the final value of $P^{[i]}(t_f)=S$ is known solution is carried out backwards in time and finally the corresponding optimal state trajectory becomes the solution of the differential equation

$$\dot{x}^{[i]} = \left[A\left(x^{[i-1]}\right) + B\left(x^{[i-1]},u^{[i-1]}\right)R^{-1}B^{T}\left(x^{[i-1]},u^{[i-1]}\right)P^{[i]}\right]x^{[i]}. \tag{11}$$

Application to Beam Model

Assuming the shaft has a uniform structure and there is no deflection in the axial direction Timoshenko beam given in Eq. (1) can be used directly. Since the beam is defined by two cross coupled PDEs in two axes they are required to be discretised in order to obtain a suitable form of equations for the control algorithm. For this purpose we shall approach the problem by discretising the Timoshenko beam model and using standard difference formulae to develop a finite-dimensional model of the system.

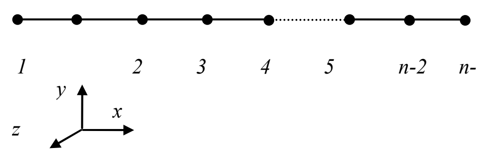

Fig. 1. Discretised beam and coordinate axes.

If we select the displacements in y- and z-axes and the time derivatives up to third order as the states, they can be defined as;

$$\xi_1^p = y,\quad \xi_2^p = \dot{y},\quad \xi_3^p = \ddot{y},\quad \xi_4^p = \dddot{y}. \tag{12}$$

$$\psi_1^p = z,\quad \psi_2^p = \dot{z},\quad \psi_3^p = \ddot{z},\quad \psi_4^p = \dddot{z}. \tag{13}$$

By defining a new state variable $\eta(t) \in \Re^{8n}$ which is formed as $[\xi(t),\psi(t)]^T$ the following form of system is obtained.

$$\dot{\eta}(t) = A(\eta(t))\eta(t) + B(\eta(t),u(t))u(t) \tag{14}$$

Control signals as mentioned before are currents which flows on the wires around the stator poles and create electromagnetic forces. Maximum and minimum values of the control current can be $\pm I_b$. Hence, when the maximum or minimum values of control current are applied no force will be generated in one side and maximum feasible force is generated in the other side.

Magnetic bearings are assumed to be located at the both ends of the shaft and electromagnetic forces are concentrated on the first and the last nodes in the discretised shaft model. And the control input vector is chosen as;

$$u(t) = [i_1^y, i_1^z, i_n^y, i_n^z]^T \tag{15}$$

Bias current for the numerical simulation is selected to be 2 A. Other values for the Timoshenko beam used in the numerical application are given in Tab. 1.

Tab. 1. Simulation Parameters.

E	207 Gpa	Ω	1000 rpm
G	77.6 Gpa	ρ	7850 kg/m^3
d	20 mm	κ	0.89

The simulation is performed in Matlab® using Euler's first order numerical integration technique. Number of nodes on the shaft is selected to be 10 in order both to get a sufficiently accurate result and to reduce the time spent for the simulation process. fig. 2 shows the displacement changes of each node on the shaft in terms of milimeters with respect to time. For simplicity only the displacements in y-axis are given.

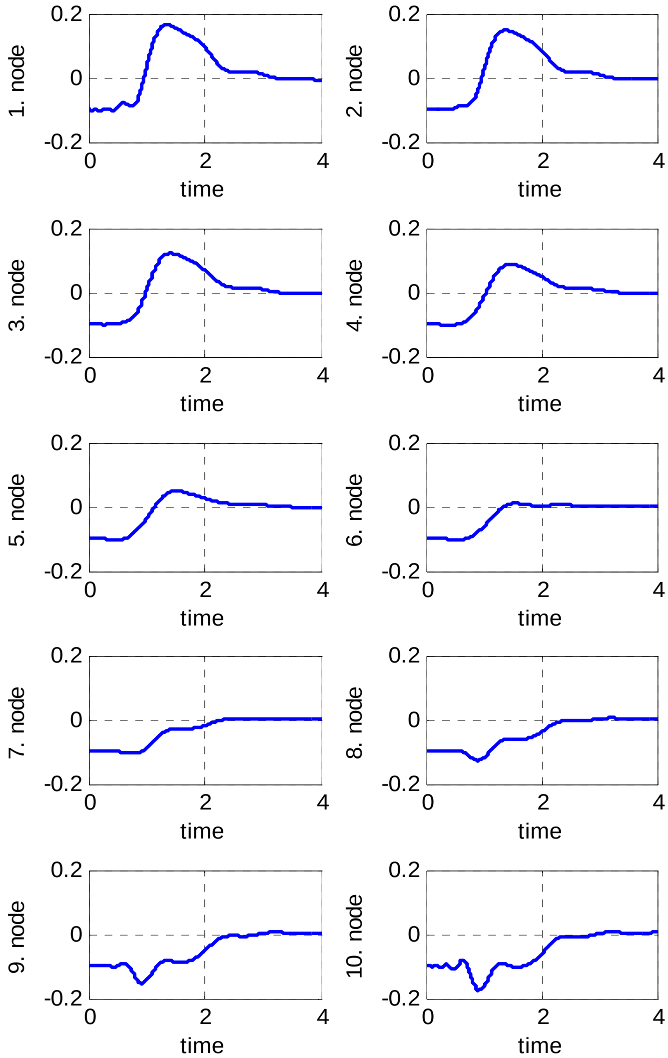

Fig. 2. Displacement of each node in y-axis.

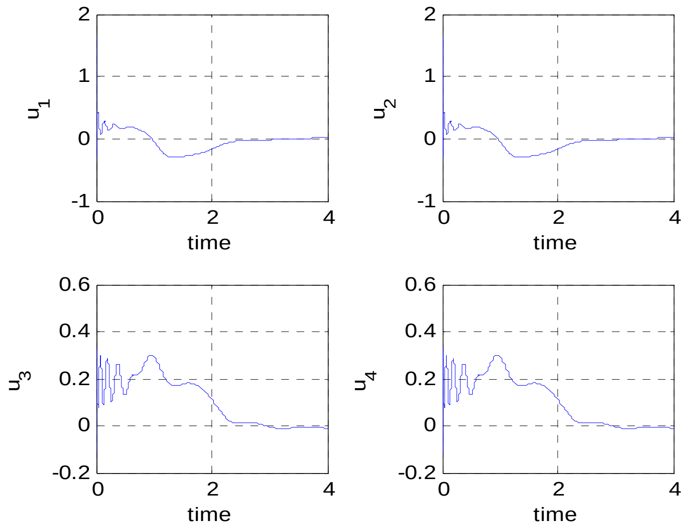

Fig. 3. Control currents.

Weighting matrices Q and R are selected to control the system in a compact time without using too much control effort. Since the bias current is selected as 2A, R is chosen to satisfy the maximum and minimum conditions for control signals. For simplicity only the response of nonlinear system to the control given as fig. 3 obtained from the sequence of LTV systems is given.

Conclusions

An approximation technique is suggested here to design a controller for nonlinear systems. Flexible shaft-AMB system is considered as the non-affine nonlinear system which is transformed to a sequence of LTV systems that enable us to design a linear controller for this sequence and it is shown that the sequence converges to the nonlinear system.

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